Gradually Varied Flow in a Trapezoidal Channel

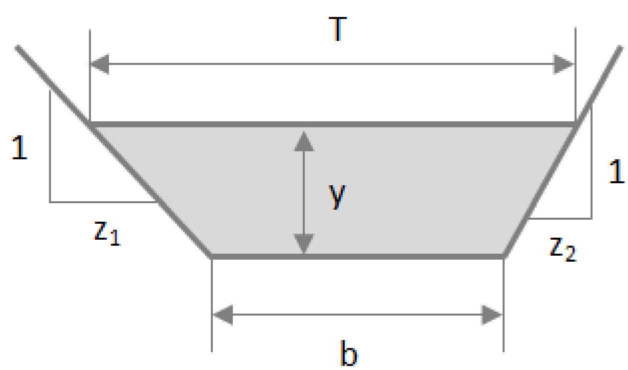
Water flows along a gently sloped trapezoidal channel with a known initial depth. As the flow progresses along the channel, the water depth eventually reaches a uniform depth that no longer changes with distance along the channel. This is known as the normal depth.

This is known as gradually varied flow. The governing differential equation is

$$\frac{d}{dx}(y) = \frac{S_0 - S_f}{1 - Fr^2}$$

where S_0 is the channel slope, S_f is the energy gradient (or the rate at which energy is lost via friction) as given by the Manning formula, Fr is the Froude number, and x is the distance along the channel.

The normal depth (i.e. when y(x) no longer changes with x) is given by S_0 - S_f



Given a trapezoidal channel, this application derives formulae for S_f and Fr. These are substituted into the differential equation, which is then numerically solved. Finally, the water surface profile along the channel is plotted.

Trapezoidal channels have several advantages over rectangular channels. Primarily, the wetted perimeter is small compared to the flow area; this reduces energy losses due to viscous drag, and thus maximizes flow.

Theory

Width of water surface

$$T := b + y(x) \cdot (z_1 + z_2)$$

Cross-sectional area of flow

$$A := \frac{y(x)}{2} \cdot (b + T) = 0.500 \cdot y(x) \cdot (2 \cdot b + y(x) \cdot (z_1 + z_2))$$

Wetted perimeter

$$P := b + y(x) \cdot \left(\sqrt{1 + z_1^2} \cdot \sqrt{1 + z_2^2}\right)$$

Hydraulic radius

$$H := \frac{A}{P} = \frac{0.500 \cdot y(x) \cdot (2 \cdot b + y(x) \cdot (z_1 + z_2))}{b + y(x) \cdot \sqrt{z_1^2 + 1} \cdot \sqrt{z_2^2 + 1}}$$

Water velocity

$$v := \frac{Q}{A} = \frac{2.000 \cdot Q}{y(x) \cdot (2 \cdot b + y(x) \cdot (z_1 + z_2))}$$

Froude number

$$Fr := \frac{v}{\sqrt{g \cdot y(x)}} = \frac{2.000 \cdot Q}{y(x) \cdot (2 \cdot b + y(x) \cdot (z_1 + z_2)) \cdot \sqrt{g \cdot y(x)}}$$

Slope of the energy gradient

$$S_f := \left(\frac{n \cdot v}{u \cdot H^{\frac{2}{3}}}\right)^2$$

$$S_f = \frac{10.079 \cdot n^2 \cdot Q^2}{y(x)^2 \cdot (2 \cdot b + y(x) \cdot (z_1 + z_2))^2 \cdot u^2 \cdot \left(\frac{y(x) \cdot (2 \cdot b + y(x) \cdot (z_1 + z_2))}{b + y(x) \cdot \sqrt{z_1^2 + 1} \cdot \sqrt{z_2^2 + 1}}\right)^{4/3}}$$

Hence the complete differential equation is

$$de := \frac{d}{dx} y(x) = \frac{S_0 - S_f}{1 - Fr^2}$$

Parameters

Bottom width and slope of channel sides

$b := 3$ $z_1 := 2$ $z_2 := 3$

Channel slope and roughness

$S_0 := 0.001$ $n := 0.025$

Coefficient in Manning equation

$u := 1$

Gravitational constant

$g := 9.81$

Flowrate

$Q := 0.2$

Maximum channel length

$L := 1000$

Water depth at maximum channel length. This will be the boundary condition on the differential equation.

$y_0 := 0.8$

The differential equation reduces to

$$de = \frac{d}{dx}\,y(x) = \frac{0.001 - \dfrac{2.520 \times 10^{-4}}{y(x)^2 \cdot (6 + 5 \cdot y(x))^2 \cdot \left(\dfrac{y(x) \cdot (6 + 5 \cdot y(x))}{3 + 7.071 \cdot y(x)}\right)^{4/3}}}{1 - \dfrac{0.016}{y(x)^3 \cdot (6 + 5 \cdot y(x))^2}}$$

Critical Depth and Normal Depth

The relative values of y_n and y_c determine the flow profile.

Normal depth

$y_n := \text{fsolve}\left(\text{eval}\left(S_f = S_0, y(x) = y\right), y = 5\right) = 0.171$

Critical depth

$y_c := \text{fsolve}\left(\text{subs}\left(y(x) = y, Fr = 1\right), y = 5\right) = 0.074$

Numerical Solution of the Differential Equation

$res := \text{dsolve}\left(\left\{de, y(L) = y_0\right\}, \text{numeric}\right)$

plots:-odeplot(res, x = 0 ..L, color = "DarkGrey", thickness = 4, view = [default, 0 ..1],
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